A Recursive Approximation Approach of non-iid Lognormal Random Variables Summation in Cellular Systems

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***Abstract*—Co-channel interference is a major factor in limiting the capacity and link quality in cellular communications. As the co-channel interference is modeled by lognormal distribution, sum of the co-channel interferences of neighboring cells is represented by the sum of lognormal Random Variables (RVs) which has no closed-form expression. Assuming independent, identically distributed (iid) RVs, the sum of lognormal RVs has been approximated by another log-normally distributed RV in the literature. In practice, the co-channel interference RVs have identical standard deviations (SDs) and different means. In this paper, first a new method based on curve fitting is proposed to approximate the sum of two log-normally distributed RV's with identical SDs and different means. Then a recursive method using the surface fitting is developed for approximating the sum of more than two lognormal RVs. Results show that the proposed method can approximate the first and the second moments of the resulting RV very well.**

***Index Terms*—Approximation, Co-Channel Interference, Cumulative Distribution Function (CDF), Curve Fitting, Lognormal Distribution, Recursive, Surface Fitting**

I. Introduction

IN a wireless system with co-channel interference, multiple non-identical shadowing processes are sensed at the receiver, due to the different obstacles in the interfering signal path. Shadowing causes attenuation which is often modeled by a lognormal distribution [1], [2]. Accordingly, sum of lognormal random variables (RVs) occurs widely in the wireless system analysis where the total co-channel interference is dealt, such as, outage probability of signal to interference and noise ratio (SINR) computation, and performance evaluation of smart antennas.

Some research has been performed in order to approximate the lognormal sum distributions [3-10]. Most of the proposed methods approximate the lognormal sum distribution with another lognormal RV [3, 8, 10]. Due to central limit theorem, this approximation is reasonable in cellular systems, where the number of interfering components with lognormal distribution is limited to the number of neighboring cells with the same frequency. In other words, the number of lognormal random variables does not approach to infinity. Fenton-Wilkinson [10], Schwartz-Yeh [10], Farley [10], and Beaulieu-Xie [3] methods are based on this approximation. To discriminate the methods, they are recalled by their renowned proposers hereafter in this paper.

Fenton-Wilkinson method matches the first and the second central moments of RVs in order to compute the mean and the standard deviation (SD) of the lognormal sum distribution. The method breaks down in computing the first and the second moments when SD of the summands are more than 4 dB [10, 11]. Therefore, Fenton-Wilkinson method cannot compute these moments accurately in cellular radio where the depth of shadowing, SD, typically ranges from 6 to 12 dB [2]. In contrast, Schwartz-Yeh method performs very well in a wide range of SDs including this range [11]. The method uses moments of the RVs in the log-domain and is much more complex than the straight forward Fenton-Wilkinson method. Schleher method, a modified version of Fenton-Wilkinson method, is based on matching successively the first and the second, the second and the third, and the third and the fourth cumulants of the reference distribution against the cumulants of the sum distribution [12]. Beaulieu-Xie method maps the CDF of the log-normal sum in a probability paper with the abscissa is in log basis (log-probability paper). With this mapping, a lognormal CDF is sketched as a straight line. Using a minimax approximation, the optimal lognormal approximation to lognormal sum is achieved. The probability paper magnifies the large and the small values of the probability; hence this method only approximates the head and the tail of the CDF. Rajwani and Beaulieu [9] apply a curve fitting technique, on the CDF of the lognormal sum, which approximates the CDF very well. The summands are assumed either independent and identically distributed (iid) RVs or non-iid RVs. However, as the curve fitting parameters depend on the statistical characteristics of the non-iid RVs, and due to the diversity of these RVs, the computation is infeasible in practice. Assuming iid summands, Lian and Ding [6] apply a quadratic polynomial curve fitting technique on the PDF of the lognormal sum. The coefficients of the polynomial are determined based on the SD and the number of

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summands. Mehta et al [8] deploy Gauss-Hermite expansion of the moment generation function to approximate the lognormal sum. Although the method has less complexity than Schwartz-Yeh method, but it is still complex compared to Fenton-Wilkinson, Beaulieu-Xie, Rajwani-Beaulieu, and Lian-Ding methods.

In practice, the co-channel interference RVs, in cellular system, have identical standard deviations (SDs) and different means. Therefore, the aforementioned methods which assume iid summands cannot be deployed for lognormal sum approximation of co-channel interference. Although Schwartz-Yeh and Mehta methods can be applied to the non-iid cases, but they are inefficient to be used in simulation due to their high complexity [8, 11]. In this paper, first a new method is proposed to approximate the sum of two log-normally distributed RVs with identical SDs and different means. Then, a recursive method, using the surface fitting, is developed for approximating the sum of more than two non-iid (hereafter we mean RVs with identical SDs and different means) lognormal RVs. To evaluate the accuracy of the proposed method, the CDF of the lognormal sum is sketched and compared with the ones of the Fenton-Wilkinson, Schwartz-Yeh, Rajwani-Beaulieu, and Beaulieu-Xie methods. The results show that the proposed method follows the CDF curve of the reference curve, achieved by Monte Carlo simulation, very well. Moreover, our proposed method is advantageous in the sense that its computational complexity is low because it just needs to evaluate 2(N-1) polynomials to find the first and the second moments of summation of N log-normally distributed RVs.

The remaining of the paper is organized as follows. The mathematical representation of the problem is brought in section II. The proposed approximation method is explained in Section III. Performance evaluation of the proposed method is represented in Section III. The paper is concluded in Section IV.

II. Problem Definition

Assume fast fading effects are averaged out, and each interfering signal has lognormal distribution. Consider N co-channel interfering signals sensed at the receiver. The *i*th received signal can be represented, in decibel [2], as follows

$$X_i = 10\log_{10}^{I_i} = \mu_i + \chi_i \ , \quad (1)$$

where μ_i is the mean power of the *i*th interfering signal, and χ_i is a zero mean normally distributed RV with SD σ_i . Accordingly, the total interference, the sum of N incoherent interfering signals, is represented as

$$I = \sum_{i=1}^{N} I_i \ . \quad (2)$$

We define *X*, the expression of *I* in decibel, in (3).

$$X = 10\log_{10}^{I} = 10\log_{10}^{(10^{X_1/10}+10^{X_2/10}+...+10^{X_N/10})} \quad (3)$$

It is well assumed that the summation of I_i s, i.e., I , follows lognormal distribution [10-12]. Therefore, X is normally distributed with mean μ and SD σ . In (3), X_i s are normally distributed RVs with mean μ_i and SD σ_i , where μ_i and σ_i are represented in decibel.

III. Proposed Method

The proposed method is explained in this section. First, the statistical characteristics of the summation of two lognormal RVs with identical SD and different means are derived in subsection *A*. Then, the method is developed for more than two lognormal RVs in subsection *B*.

A. Summation of Two Lognormal RVs with Identical SDs

Consider two log-normally distributed interfering signals, X_1 and X_2 , with identical SDs, σ dB, and different means, μ_1 and μ_2 dB. The corresponding summation of X_1 and X_2 in log domain, X , is

$$X = 10\log_{10}^{I} = 10\log_{10}^{(10^{X_1/10}+10^{X_2/10})} . \quad (4)$$

By extracting the mean value of X_1, i.e. μ_1, X is expressed as

$$X = \mu_1 + 10\log_{10}^{(10^{X_1-\mu_1/10}+10^{X_2-\mu_1/10})} \quad (5)$$

where $X_1-\mu_1$ and $X_2-\mu_1$ are RVs with zero and $\Delta\mu = \mu_2-\mu_1$ means, respectively.

Defining Y as (6), we represent X , μ_X , and σ_X in terms of Y , mean and SD of Y denoted by μ_Y , and σ_Y , respectively, in (7) to (9).

$$Y = 10\log_{10}^{(10^{X_1-\mu_1/10}+10^{X_2-\mu_2/10})} \quad (6)$$

$$X = \mu_1 + Y \quad (7)$$

$$\mu_X = \mu_1 + \mu_Y \quad (8)$$

$$\sigma_X = \sigma_Y \quad (9)$$

Accordingly, to find the first and the second moments of *X*, we need to find the corresponding moments of *Y*. Thus, the problem turns to identifying the mean of the sum of two lognormal random variables, $10^{X_1-\mu_1/10}$ and $10^{X_2-\mu_1/10}$, with zero and $\Delta\mu = \mu_2 - \mu_1$ means, respectively. Thus, the mean and SD of Y, i.e., μ_Y and σ_Y , for a specific σ are functions of $\Delta\mu$. These functions can be accessible by Monte-Carlo simulation for a range of $\Delta\mu$.

Then we draw the fitting curves along the mean and the SD curves of the sum distribution in Fig. 1 and Fig. 2, respectively, for σ 6, 8 and 12 dB. Fitting is performed with the functions $\overline{\mu}_Y$ and $\overline{\sigma}_Y$, represented in (10) and (11), respectively, and whose parameters are given in Table I. The fitting functions are "good" fits for the simulation points μ_Y and σ_Y . The coefficient determination, i.e., R^2 , which is an indicator of the fit perfection is used to represent the goodness of fitting. For the best fitness, $R^2 = 1$, and $R^2 < 1$ represents imperfect fitness. The coefficient determination is over 0.99 for $\overline{\mu}_Y$ and $\overline{\sigma}_Y$,

$$\overline{\mu}_Y\,(\Delta\mu) = \alpha e^{-\left(\frac{\Delta\mu - b}{c}\right)^2} \tag{10}$$

$$\overline{\sigma}_Y\,(\Delta\mu) = \sigma - d e^{-\left(\frac{\Delta\mu}{g}\right)^2} \tag{11}$$

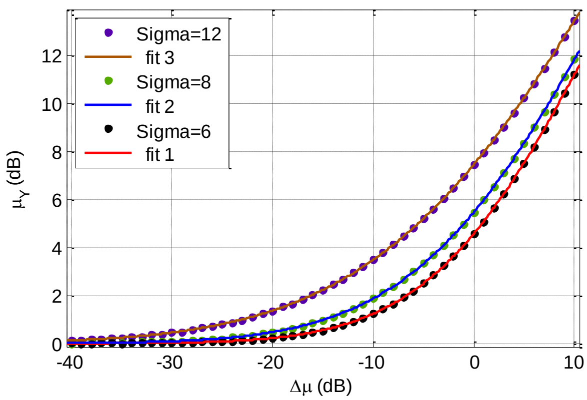

Fig. 1. The mean curves of the simulated RVs along with the fitted curves for 3 values of σ

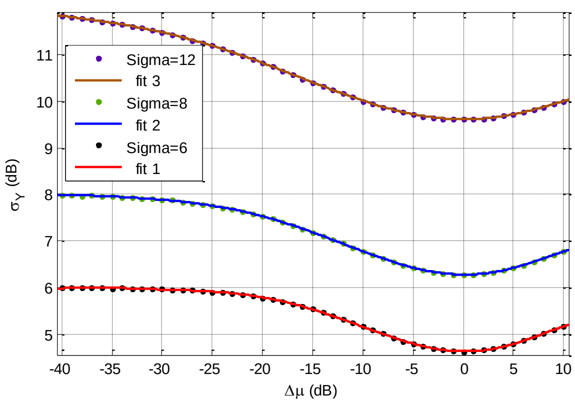

Fig. 2. The standard deviation curves of the simulated RVs along the fitted curves for 3 values of σ

TABLE I

CURVE FITTING FUNCTION PARAMETERS FOR 3 VALUES OF σ

σ	α	b	c	d	g
6	20.23	27.03	22.15	1.364	14.61
8	22.51	30.65	25.77	1.708	17.61
12	28.18	39.32	34.11	2.371	24.07

B. Summation of More Than Two Lognormal RVs with Identical SDs

Considering the sum of N log-normally distributed interfering signals with identical SDs and different means, we propose a recursive approach to approximate the first and the second moments of the sum distribution. In the first step of the recursive approach, the first and the second moments of the sum of two interfering signals are computed. In the next step, the first and the second moments of the resulting RV and the next interfering signal are computed. Since the summation of two interfering signals with identical SDs, i.e. σ , has a different SD ($\sigma_r \neq \sigma$), a recursive approach using the functions (9) and (10), is not feasible. This motivated us to extend the expressed method in subsection *A* as follows. Assume two RVs with (0, σ) and ($\Delta\mu$, σ_r). In the recursive approach, one summand should always have a constant zero mean and the SD σ . Adding another dimension, σ_r , to the curves of the previous subsection changes them to surfaces. Using a Monte-Carlo simulation followed by surface fitting, we can obtain the mean and the SD of the sum distribution as a function of $\Delta\mu$ and σ_r . Fig. 3 and Fig. 4 show the simulation results and fitting curves for $\sigma = 12$. Fitting is performed with the functions $\overline{\overline{\mu}}_Y$ and $\overline{\overline{\sigma}}_Y$, represented by (12) and (13), fitting the simulation points, μ_Y and σ_Y , respectively. The coefficient determination is over 0.99 for $\overline{\overline{\mu}}_Y$ and $\overline{\overline{\sigma}}_Y$.

$$\overline{\overline{\mu}}_Y\,(\Delta\mu, \sigma_r) = \sum\sum q_{ij}\,\Delta\mu^i\,\sigma_r{}^j \tag{12}$$

Where $j \le 3, i \in \{0,1,2,3\}, j \in \{0,1\}$

$$\overline{\overline{\sigma}}_Y\,(\Delta\mu, \sigma_r) = \sum\sum p_{ij}\,\Delta\mu^i\,\sigma_r{}^j \tag{13}$$

Where $i + j \le 5, i \in \{0,1,...,5\}, j \in \{0,1\}$

The parameters of the fitting functions in (12) and (13) are listed in Table II.

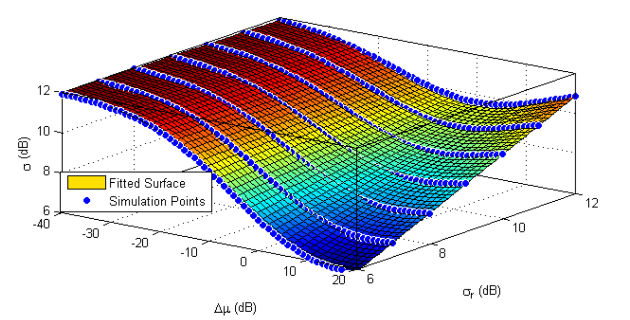

Fig. 3. The standard deviation of the summation of two log-normally distributed signals ($\mu_1 = 0, \sigma_1 = 12, \mu 2 = \Delta\mu, \sigma_2 = \sigma_r$)

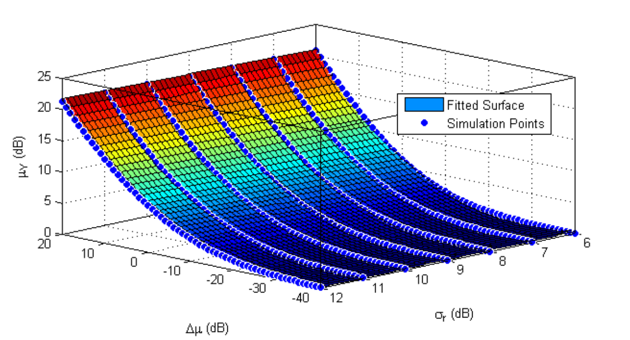

Fig. 4. The mean of the summation of two log-normally distributed signals ($\mu_1 = 0, \sigma_1 = 12, \mu 2 = \Delta\mu, \sigma_2 = \sigma_r$)

TABLE II
SURFACE FITTING FUNCTION PARAMETERS FOR 3 VALUES OF σ

σ	6	8	12
p_{00}	2.575	3.503	5.388
p_{10}	-0.2494	-0.2998	-0.3572
p_{20}	1.282e-3	1.27e-3	1.185e-3
p_{30}	2.93e-4	3.116e-4	2.751e-4
p_{40}	5.067e-7	8.964e-7	1.122e-6
p_{50}	-9.004e-8	-8.204e-8	-5.202e-8
p_{01}	0.3503	0.3476	0.3478
p_{11}	0.04094	0.03713	0.0298
p_{21}	5.765e-4	4.678e-4	2.781e-4
p_{31}	-4.1e-5	-3.382e-5	-2.117e-5
p_{41}	-9.039e-7	-7.064e-7	-3.833e-7
q_{00}	3.956	4.305	5.149
q_{10}	0.46	0.4665	0.4784
q_{20}	0.0154	0.01424	0.01208
q_{30}	1.536e-4	1.234e-4	7.498e-5
q_{01}	0.1173	0.1458	0.1858
q_{11}	-2.184e-4	-2.952e-5	4.754e-5
q_{20}	-1.16e-4	-1.307e-4	-1.377e-4

IV. PERFORMANCE EVALUATION

The performance of the proposed method in terms of the closeness of its cumulative distribution function (CDF) to the ones of the Monte Carlo, Fenton-Wilkinson, Beaulieu-Xie, Rajwani- Beaulieu, and Schwartz-Yeh is investigated in this paper. While our method is proposed for non-iid RVs, it is supposed to perform well for approximation of iid RVs too. Therefore, we conduct the simulation in subsection *A* assuming iid lognormal RVs. In subsection *B*, non-iid lognormal RVs, with identical SD but different means, are considered in the Monte-Carlo simulation.

A. Sum of iid Lognormal RVs

An approximation to the sum of six iid lognormal RVs ($\sigma = 12$ and $\mu = 0$ dB) are computed using Fenton-Wilkinson, Beaulieu-Xie, and the proposed methods. The CDFs of the random variables achieved by each method along with the one of the Monte-Carlo simulation are depicted in Fig. 5. The figure shows Fenton-Wilkinson method performs well in the tail portion of the CDF; Beaulieu-Xie method tracks the CDF in the extreme head and tail portion of the CDF. It is observed that the proposed method matches the simulation results very well, and it is a better approximation than the other two methods at most parts of the CDF.

According to the simulation, our proposed method performs as well as Rajwani-Beaulieu and Schwartz-yeh methods in approximating the sum. Therefore, to distinguish the three methods performance, we define the CDF error as the difference between the CDFs of the RV sum identified by the Monte-Carlo simulation and each method. The CDF error for Rajwani-Beaulieu, Schwartz-yeh, and the proposed methods are depicted in Fig. 6. It can be seen that the proposed method has a maximum error of 0.022 near the tail portion of the CDF. Although Schwartz-Yeh method has low error through the abscissa, its computational complexity is a limiting factor [8, 10].

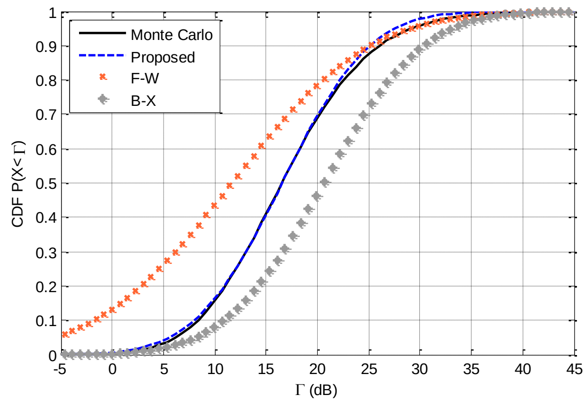

Fig. 5. The CDF of lognormal sum approximated by Fenton-Wilkinson (F-W), Beaulieu-Xie (B-X), and the proposed methods

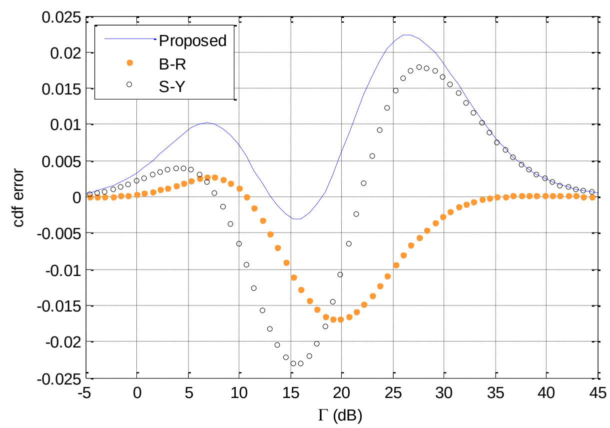

Fig. 6. The CDF error of lognormal sum approximated by Rajwani-Beaulieu (B-R), Schwartz-Yeh (S-Y), and the proposed methods

B. Sum of Lognormal RVs with Different Means

We assume the summands are non identical in terms of their means, but they have identical SD ranging from 6 to 12dB, conforming the typical SD of the co-channel interference in cellular system. Fig. 7 compares the performance of Fenton-Wilkinson, Beaulieu-Xie, and the proposed methods when approximating the sum of six lognormal RVs ($\sigma = 12$ and $\mu_{1-6} = -25, -15, -5, 5, 15, 25$). It can be seen that the proposed method tracks the CDF more accurately than the other two methods through the abscissa.

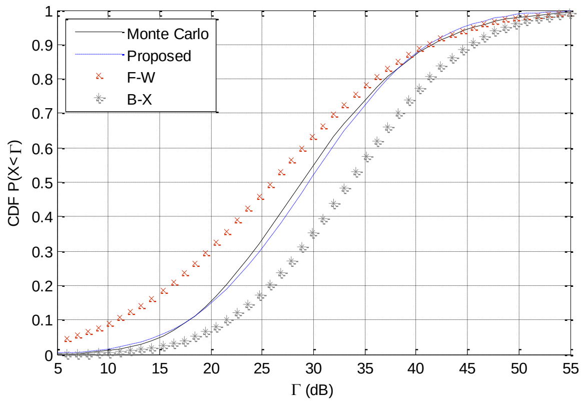

Fig. 7. The CDF of the lognormal sum, of the non-iid RVs, approximated using Fenton-Wilkinson, Beaulieu-Xie, and the proposed methods

V. CONCLUSION

A new method based on curve fitting has been proposed to approximate the lognormal sum of two lognormal RVs with different means and identical SD. The method has been extended, using the surface fitting, to approximate the lognormal sum of more than two lognormal non-iid RVs recursively. Comparing the Monte-Carlo simulation results with the ones of our method and some renown methods referred in the literature shows that the proposed method can approximate the first and the second moments of the resulting RV sum very well.

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